Error Detection
and
Correction

Note

The Hamming distance between two words is the number of differences between corresponding bits.

Example 10.4

Let us find the Hamming distance between two pairs of words.

1. The Hamming distance d(000, 011) is 2 because

$\boxed{000 \oplus 011 \text{ is } 011 \text{ (two 1s)}}$

2. The Hamming distance d(10101, 11110) is 3 because

$\boxed{10101 \oplus 11110 \text{ is } 01011 \text{ (three 1s)}}$

Note

The minimum Hamming distance is the smallest Hamming distance between all possible pairs in a set of words.

Example 10.5

Find the minimum Hamming distance of the coding scheme in Table 10.1.

Solution

We first find all Hamming distances.

$d(000, 011) = 2$	$d(000, 101) = 2$	$d(000, 110) = 2$	$d(011, 101) = 2$
$d(011, 110) = 2$	$d(101, 110) = 2$		

The d_{min} in this case is 2.

Example 10.6

Find the minimum Hamming distance of the coding scheme in Table 10.2.

Solution

We first find all the Hamming distances.

$d(00000, 01011) = 3$	$d(00000, 10101) = 3$	$d(00000, 11110) = 4$
$d(01011, 10101) = 4$	$d(01011, 11110) = 3$	$d(10101, 11110) = 3$

The d_{min} in this case is 3.

Note

To guarantee the detection of up to s errors in all cases, the minimum Hamming distance in a block code must be $d_{min} = s + 1$.

10-3 LINEAR BLOCK CODES

Almost all block codes used today belong to a subset called linear block codes. A linear block code is a code in which the exclusive OR (addition modulo-2) of two valid codewords creates another valid codeword.

Topics discussed in this section:

Minimum Distance for Linear Block Codes
Some Linear Block Codes

Note

In a linear block code, the exclusive OR (XOR) of any two valid codewords creates another valid codeword.

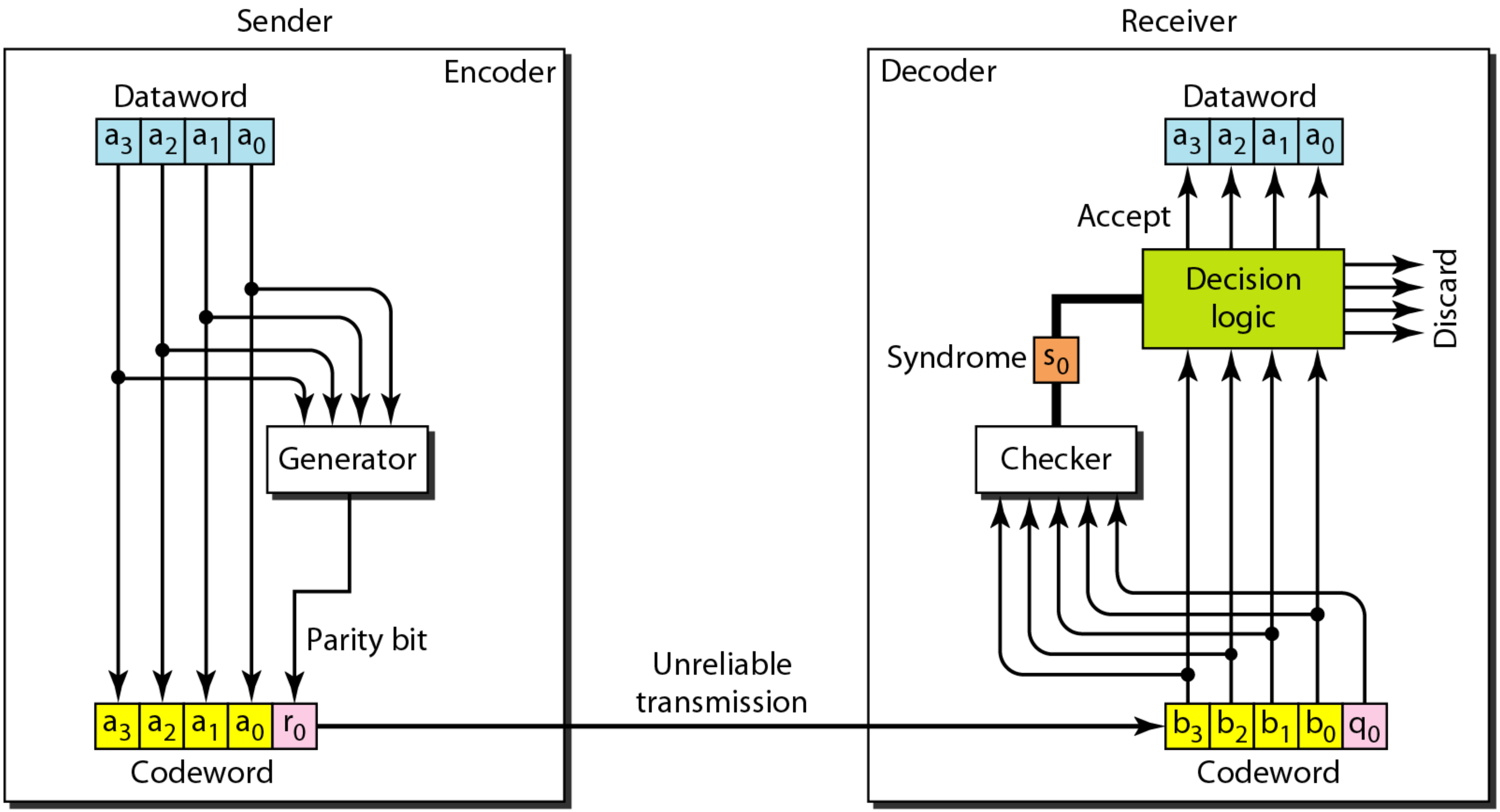
Note

A simple parity-check code is a single-bit error-detecting code in which $n = k + 1$ with $d_{min} = 2$.
Even parity (ensures that a codeword has an even number of 1's) and odd parity (ensures that there are an odd number of 1's in the codeword)

Table ***Simple parity-check code C(5, 4)***

Datawords	*Codewords*	*Datawords*	*Codewords*
0000	00000	1000	10001
0001	00011	1001	10010
0010	00101	1010	10100
0011	00110	1011	10111
0100	01001	1100	11000
0101	01010	1101	11011
0110	01100	1110	11101
0111	01111	1111	11110

Figure *Encoder and decoder for simple parity-check code*



Example

Let us look at some transmission scenarios. Assume the sender sends the dataword 1011. The codeword created from this dataword is 10111, which is sent to the receiver. We examine five cases:

1. *No error occurs; the received codeword is 10111. The syndrome is 0. The dataword 1011 is created.*
2. *One single-bit error changes a_1. The received codeword is 10011. The syndrome is 1. No dataword is created.*
3. *One single-bit error changes r_0. The received codeword is 10110. The syndrome is 1. No dataword is created.*

Example (continued)

4. An error changes r_0 and a second error changes a_3 . The received codeword is 00110. The syndrome is 0. The dataword 0011 is created at the receiver. Note that here the dataword is wrongly created due to the syndrome value.

5. Three bits—a_3, a_2, and a_1—are changed by errors. The received codeword is 01011. The syndrome is 1. The dataword is not created. This shows that the simple parity check, guaranteed to detect one single error, can also find any odd number of errors.

Note

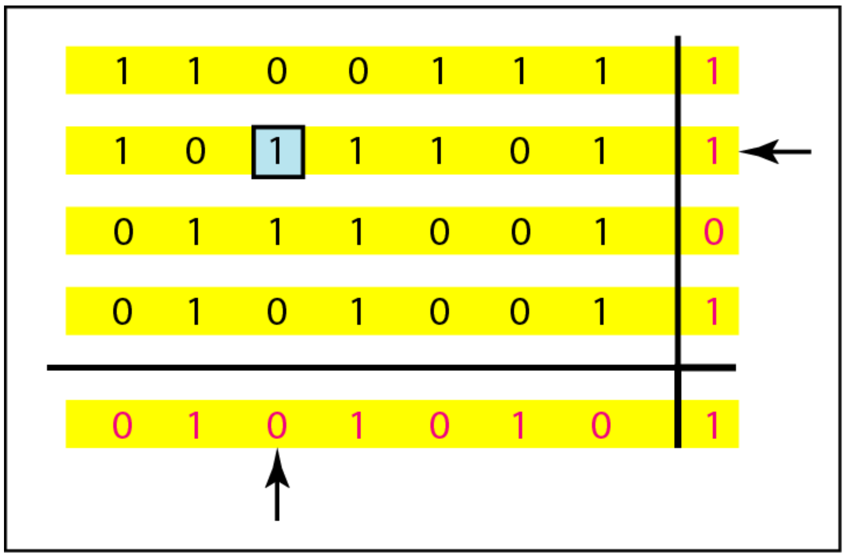
A simple parity-check code can detect an odd number of errors.

Figure *Two-dimensional parity-check code*

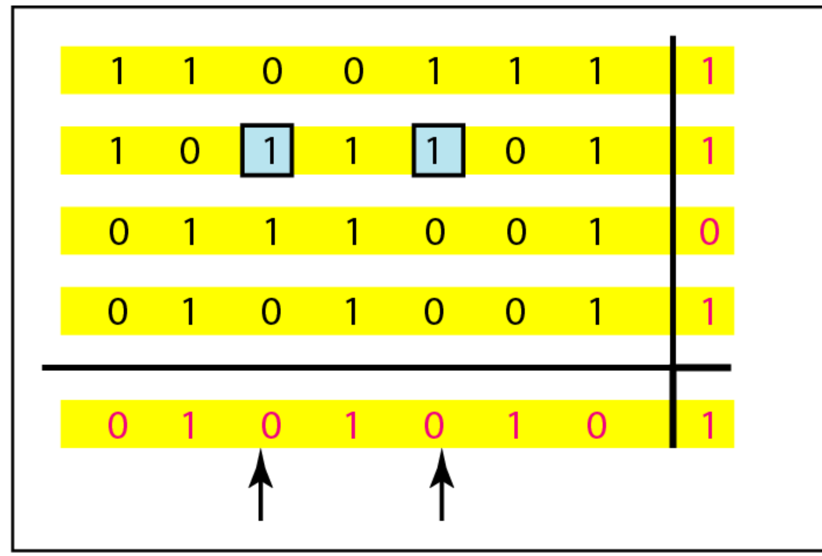
							Row parities
1	1	0	0	1	1	1	1
1	0	1	1	1	0	1	1
0	1	1	1	0	0	1	0
0	1	0	1	0	0	1	1
0	1	0	1	0	1	0	1
Column parities							

a. Design of row and column parities

Figure *Two-dimensional parity-check code*

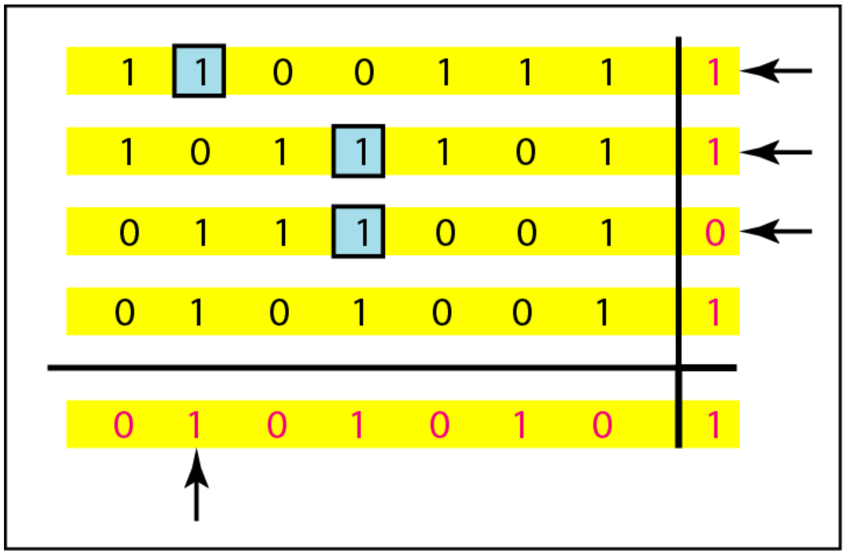


b. One error affects two parities

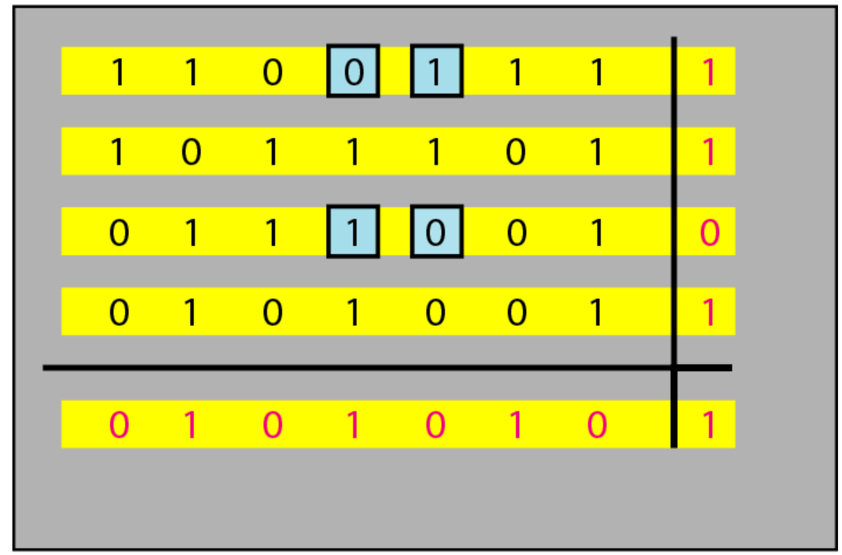


c. Two errors affect two parities

Figure *Two-dimensional parity-check code*



d. Three errors affect four parities



e. Four errors cannot be detected

Table ***Hamming code C(7, 4) - n=7, k = 4***

Datawords	*Codewords*	*Datawords*	*Codewords*
0000	0000000	1000	1000110
0001	0001101	1001	1001011
0010	0010111	1010	1010001
0011	0011010	1011	1011100
0100	0100011	1100	1100101
0101	0101110	1101	1101000
0110	0110100	1110	1110010
0111	0111001	1111	1111111

Calculating the parity bits at the transmitter
:

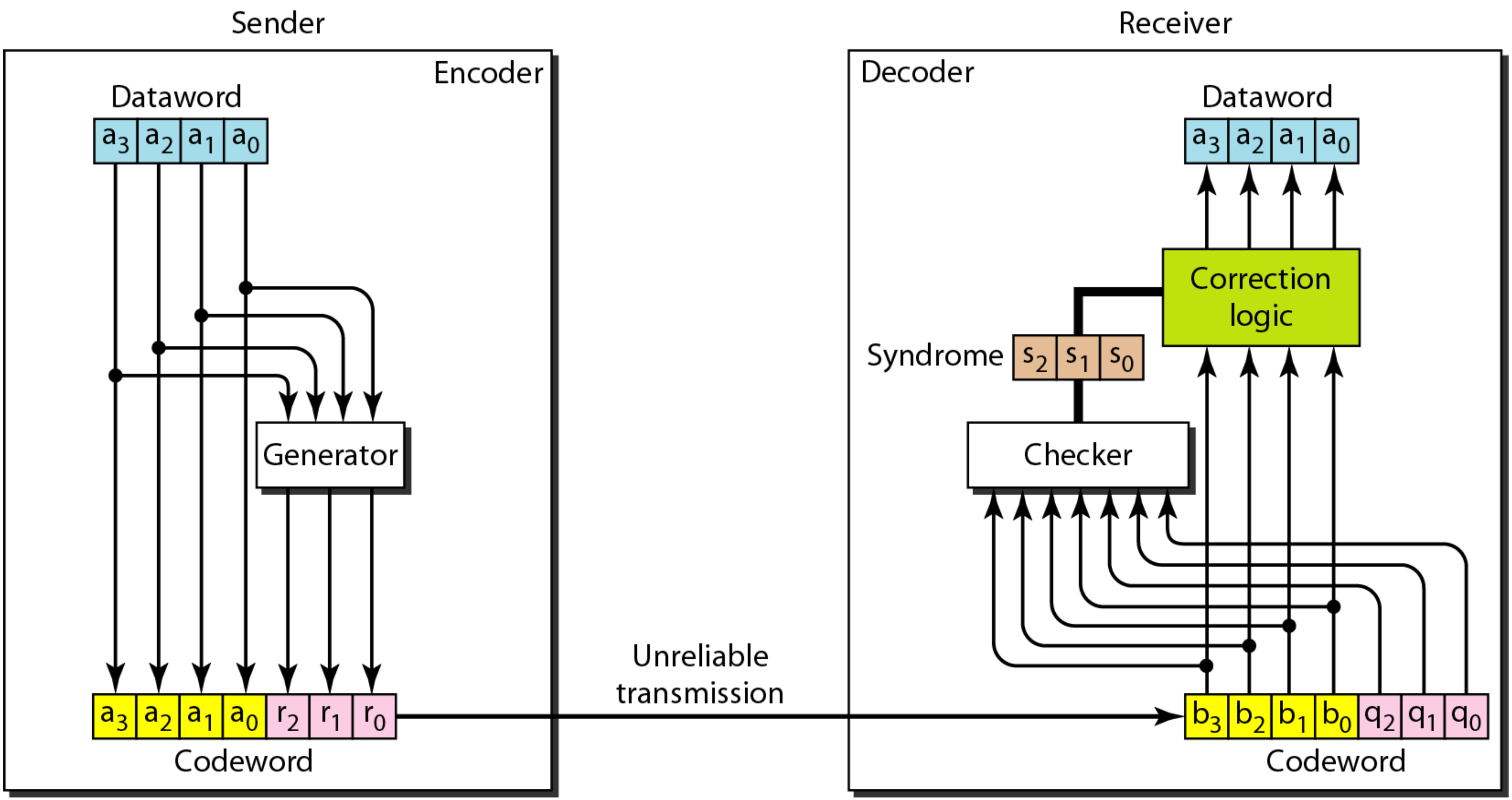
Modulo 2 arithmetic:

$r_0 = a_2 + a_1 + a_0$
$r_1 = a_3 + a_2 + a_1$
$r_2 = a_1 + a_0 + a_3$

Calculating the syndrome at the receiver:

$s_0 = b_2 + b_1 + b_0$
$s_1 = b_3 + b_2 + b_1$
$s_2 = b_1 + b_0 + b_3$

Figure ***The structure of the encoder and decoder for a Hamming code***



Burst Errors

- Burst errors are very common, in particular in wireless environments where a fade will affect a group of bits in transit. The length of the burst is dependent on the duration of the fade.
- One way to counter burst errors, is to break up a transmission into shorter words and create a block (one word per row), then have a parity check per word.
- The words are then sent column by column. When a burst error occurs, it will affect 1 bit in several words as the transmission is read back into the block format and each word is checked individually.

CYCLIC CODES

***Cyclic codes* are special linear block codes with one extra property. In a cyclic code, if a codeword is cyclically shifted (rotated), the result is another codeword.**

Table ***A CRC code with C(7, 4)***

Dataword	*Codeword*	*Dataword*	*Codeword*
0000	0000000	1000	1000101
0001	0001011	1001	1001110
0010	0010110	1010	1010011
0011	0011101	1011	1011000
0100	0100111	1100	1100010
0101	0101100	1101	1101001
0110	0110001	1110	1110100
0111	0111010	1111	1111111

Figure *CRC encoder and decoder*

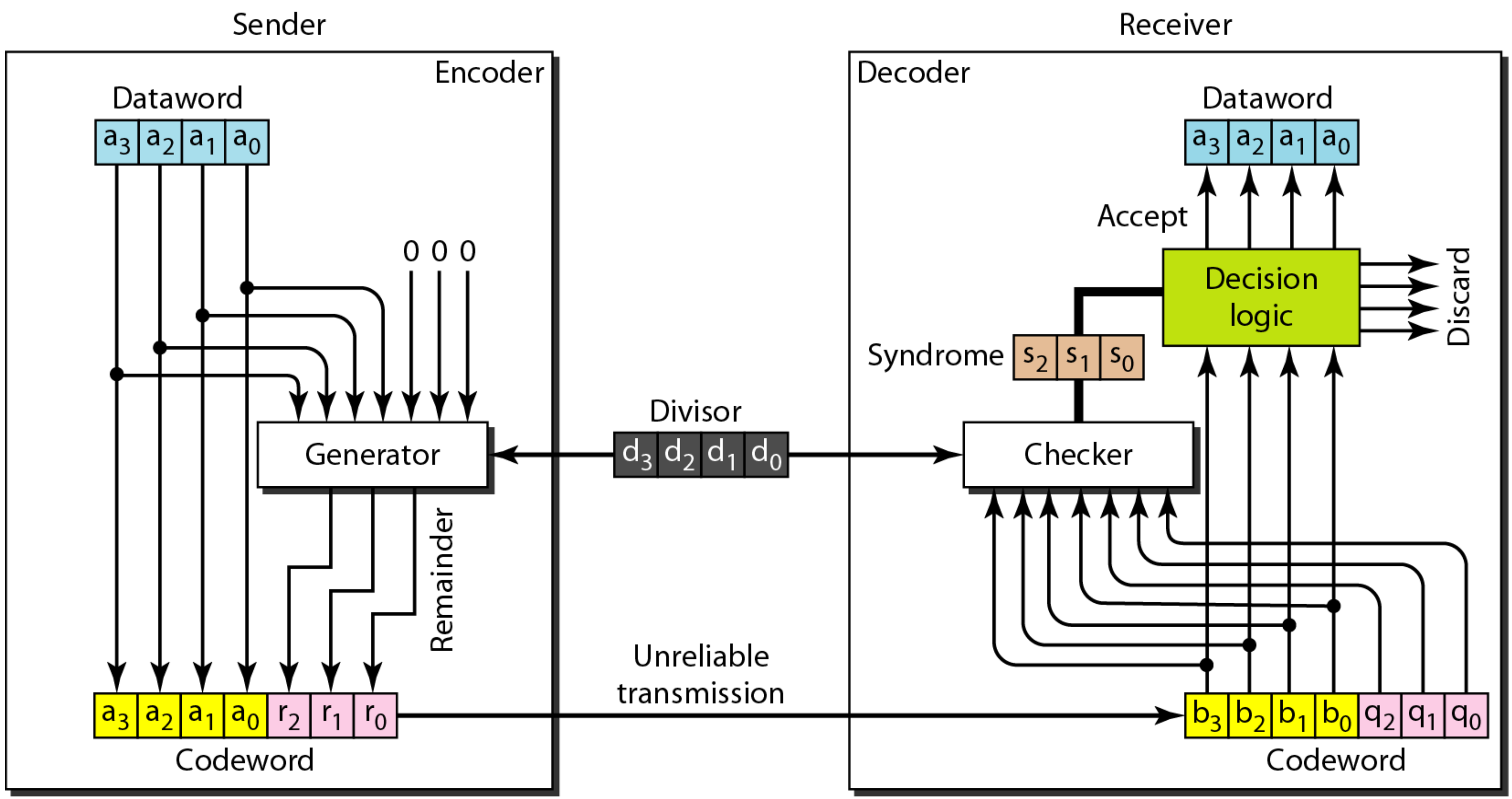


Figure *Division in CRC encoder*

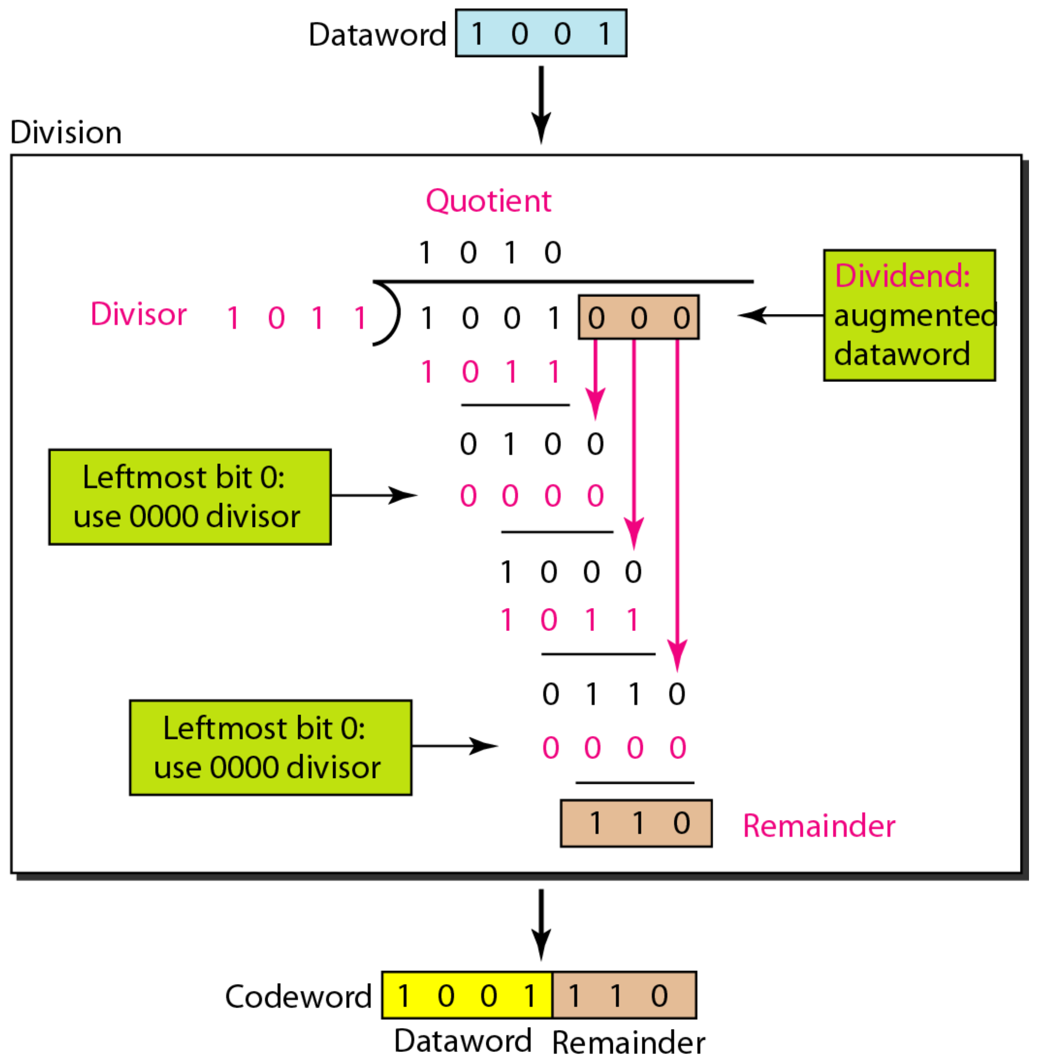


Figure *Division in the CRC decoder for two cases*

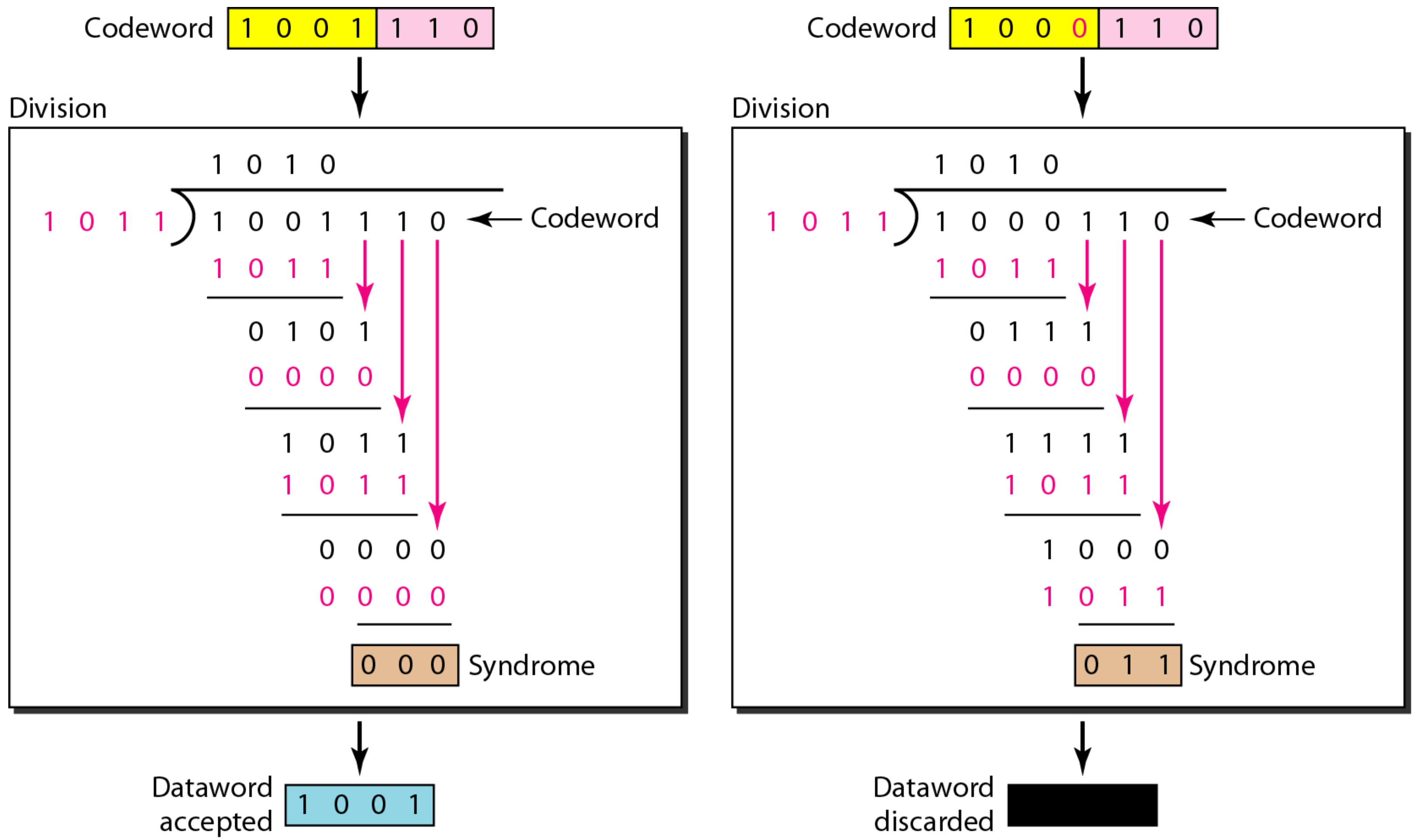


Figure *Hardwired design of the divisor in CRC*

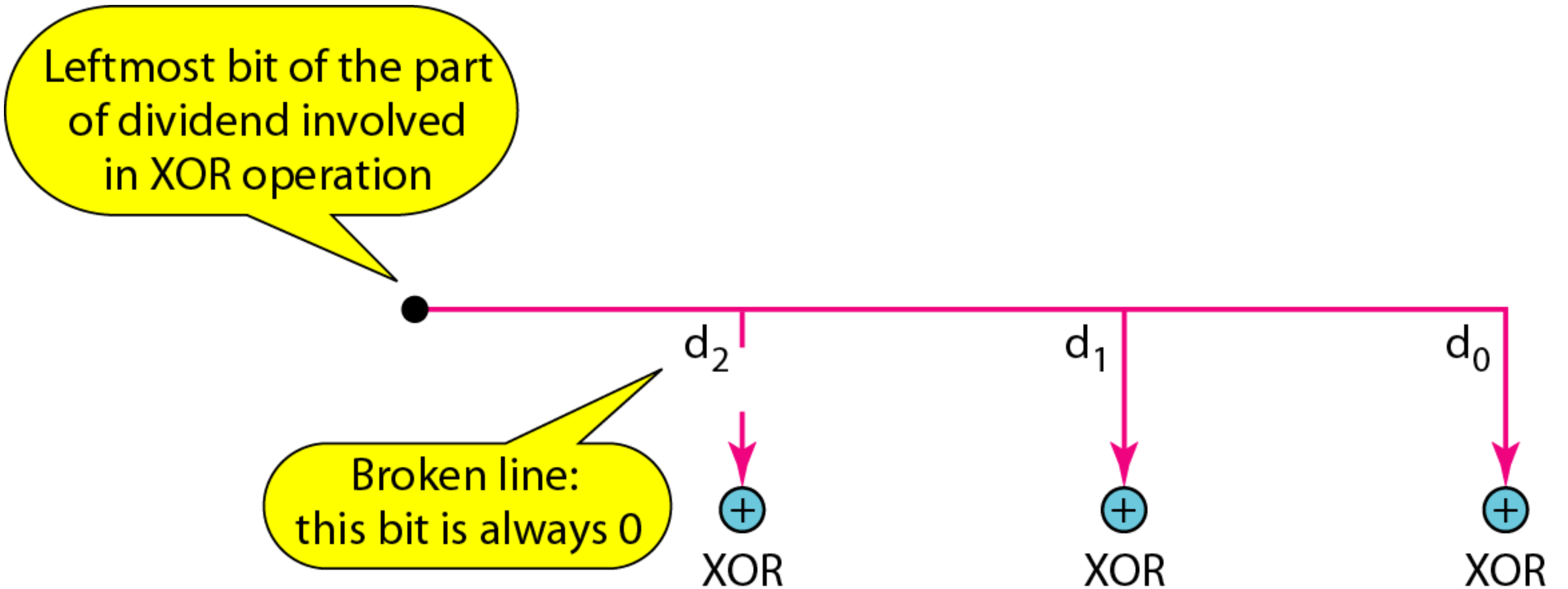


Figure *Simulation of division in CRC encoder*

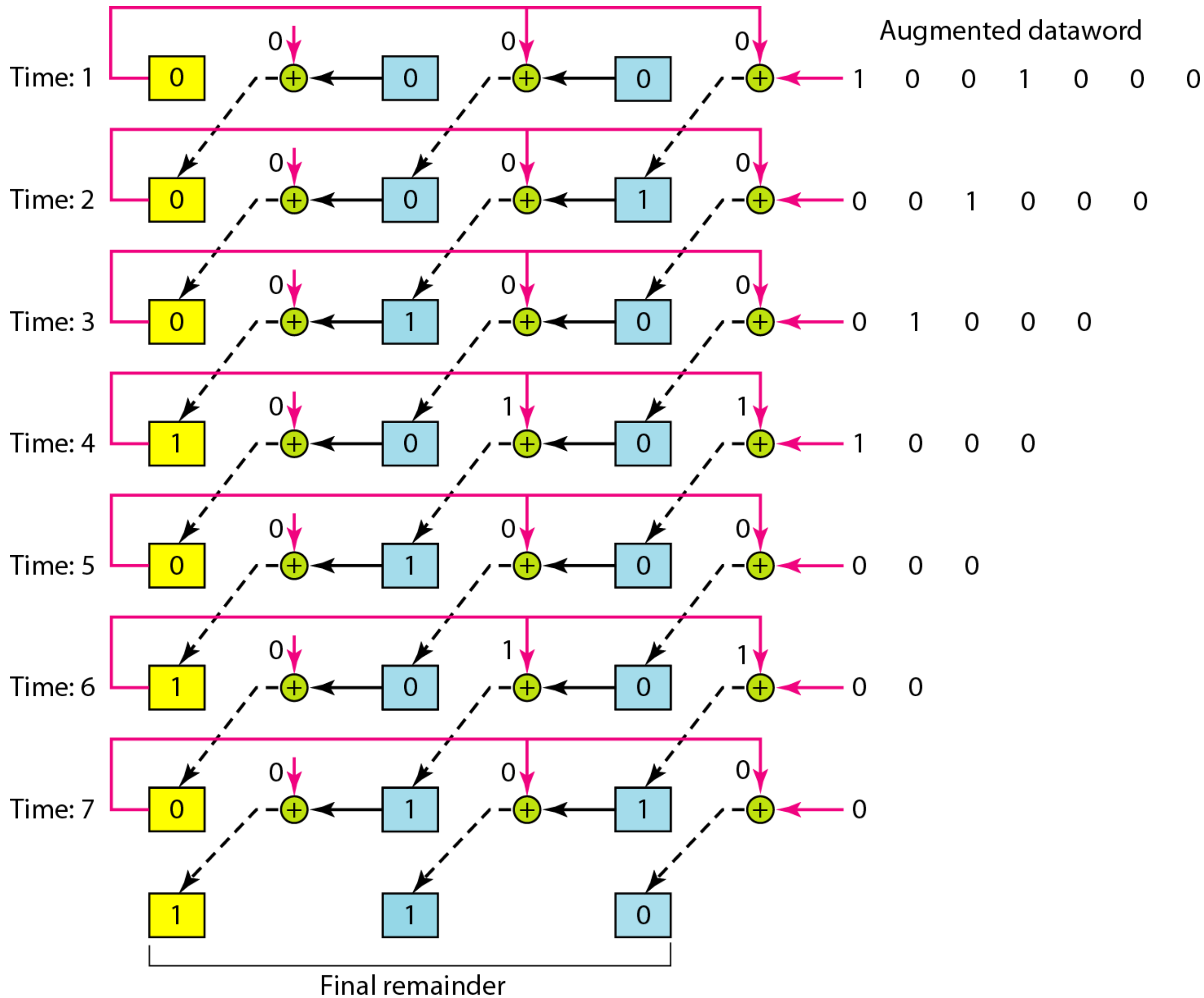


Figure *The CRC encoder design using shift registers*

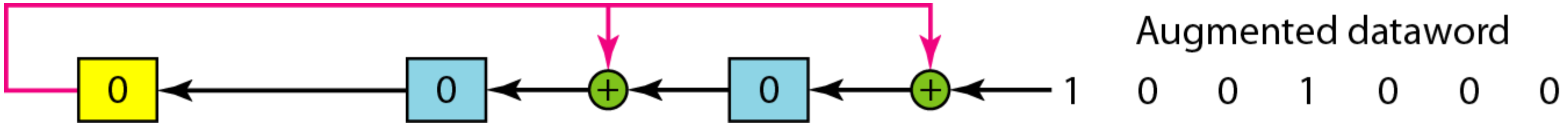
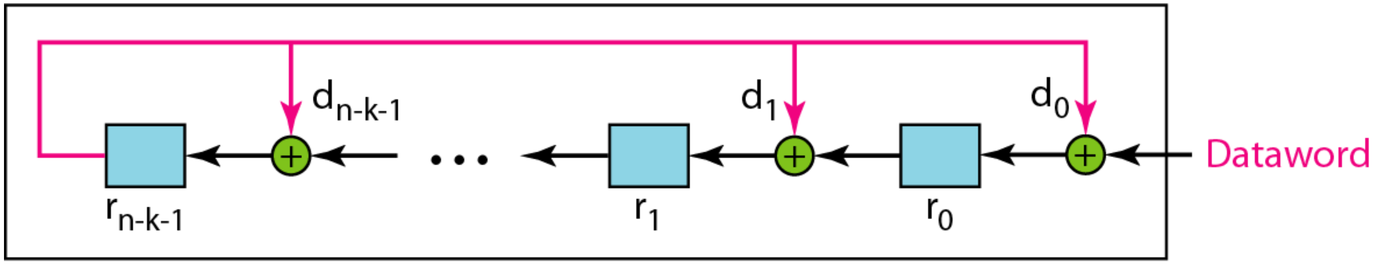


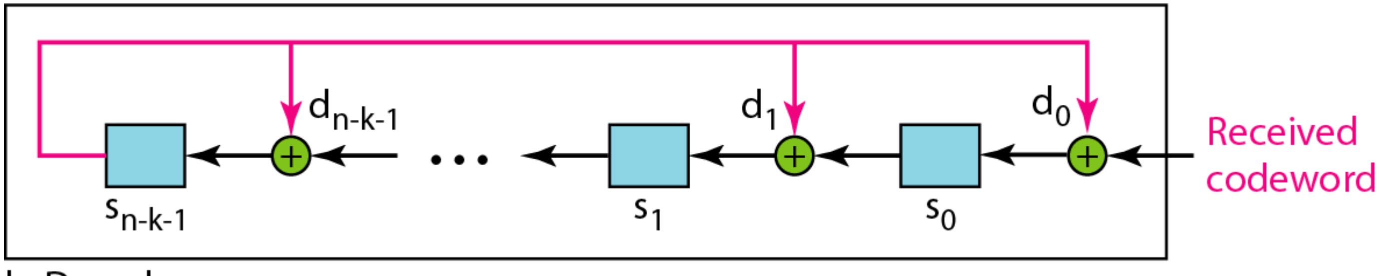
Figure *General design of encoder and decoder of a CRC code*

Note:

The divisor line and XOR are missing if the corresponding bit in the divisor is 0.



a. Encoder

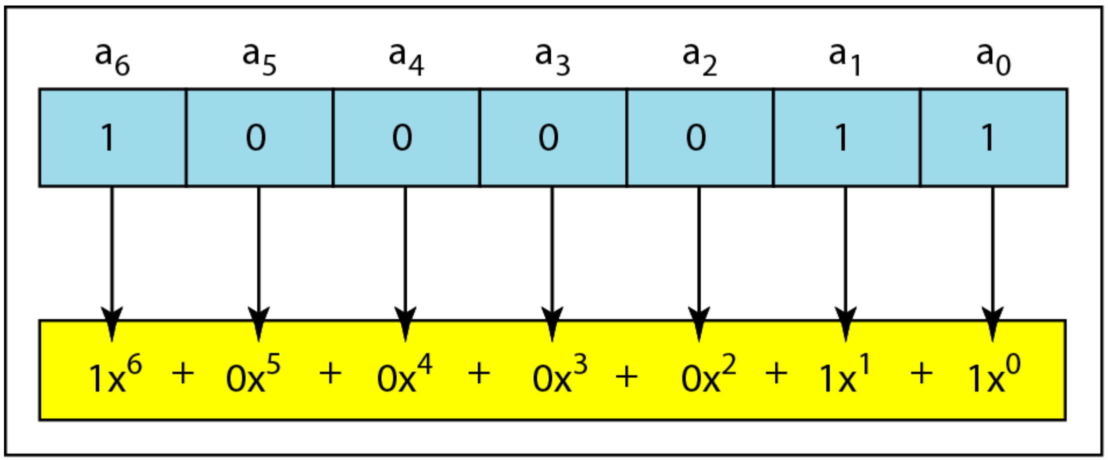


b. Decoder

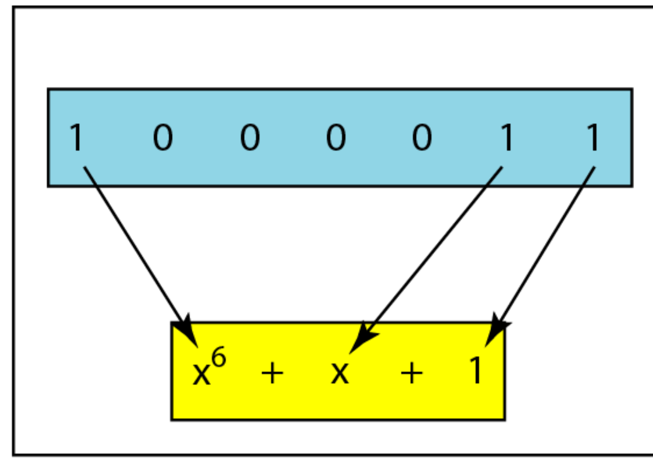
Using Polynomials

- We can use a polynomial to represent a binary word.
- Each bit from right to left is mapped onto a power term.
- The rightmost bit represents the “0” power term. The bit next to it the “1” power term, etc.
- If the bit is of value zero, the power term is deleted from the expression.

Figure *A polynomial to represent a binary word*

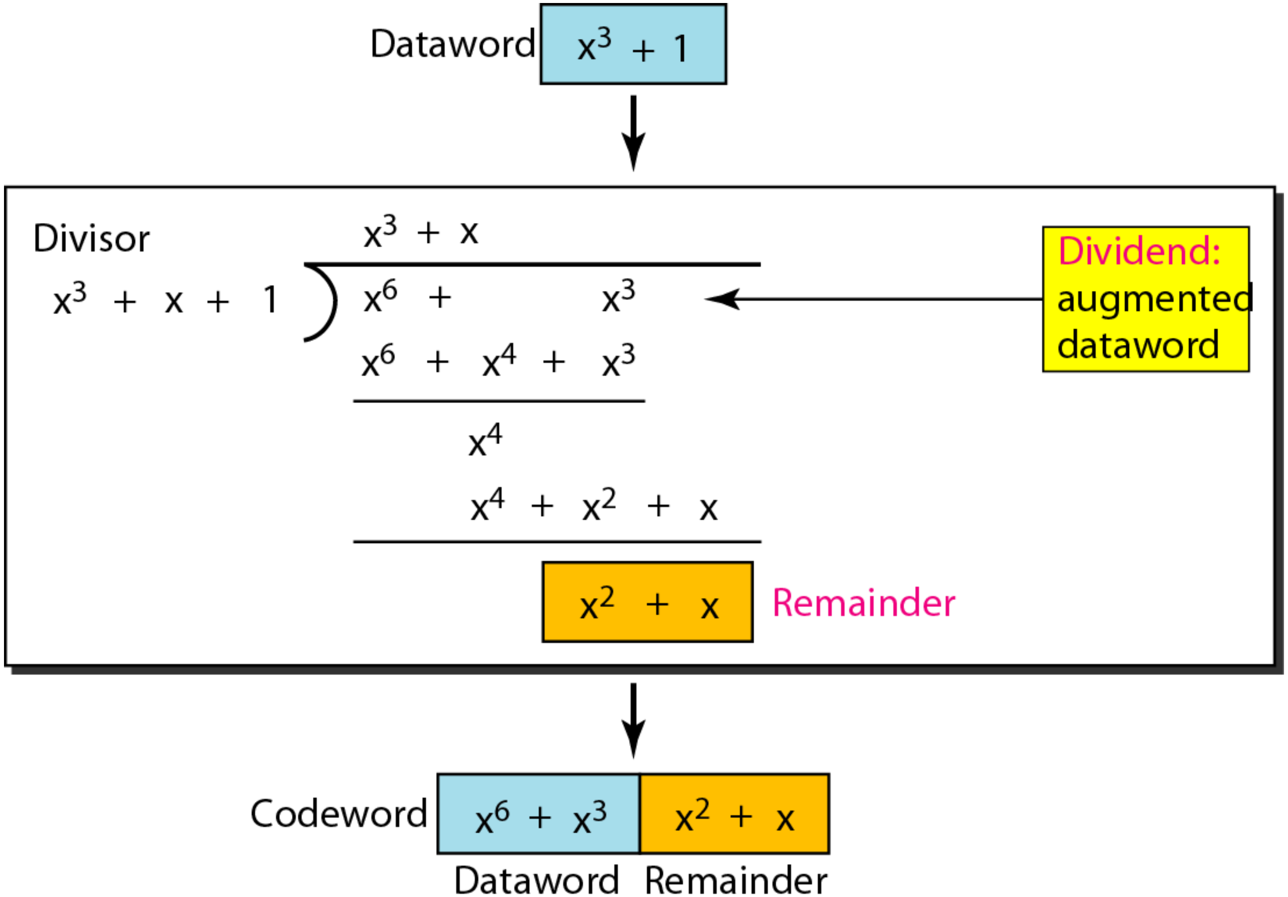


a. Binary pattern and polynomial



b. Short form

Figure *CRC division using polynomials*



Note

The divisor in a cyclic code is normally called the generator polynomial or simply the generator.

Note

In a cyclic code,

If $s(x) \neq 0$, one or more bits is corrupted.

If $s(x) = 0$, either

a. No bit is corrupted. or

b. Some bits are corrupted, but the decoder failed to detect them.

Table ***Standard polynomials***

Name	*Polynomial*	*Application*
CRC-8	$x^8 + x^2 + x + 1$	ATM header
CRC-10	$x^{10} + x^9 + x^5 + x^4 + x^2 + 1$	ATM AAL
CRC-16	$x^{16} + x^{12} + x^5 + 1$	HDLC
CRC-32	$x^{32} + x^{26} + x^{23} + x^{22} + x^{16} + x^{12} + x^{11} + x^{10} +$ $x^8 + x^7 + x^5 + x^4 + x^2 + x + 1$	LANs

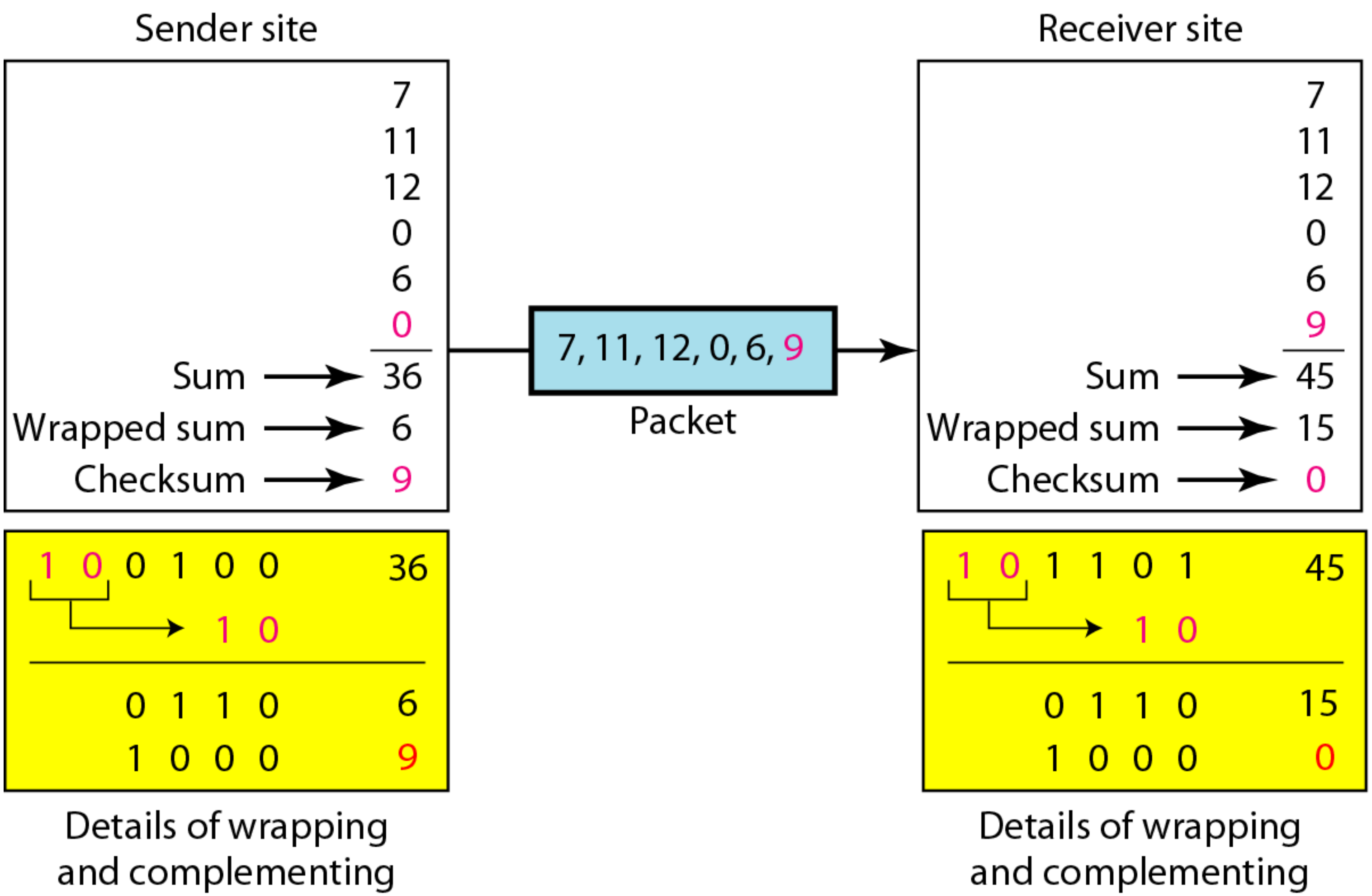
CHECKSUM

The last error detection method we discuss here is called the checksum. The checksum is used in the Internet by several protocols although not at the data link layer. However, we briefly discuss it here to complete our discussion on error checking

Topics discussed in this section:

Idea
One's Complement
Internet Checksum

Figure *Example*



Note

Sender site:

1. The message is divided into 16-bit words.
2. The value of the checksum word is set to 0.
3. All words including the checksum are added using one’s complement addition.
4. The sum is complemented and becomes the checksum.
5. The checksum is sent with the data.

Note

Receiver site:

1. **The message (including checksum) is divided into 16-bit words.**
2. **All words are added using one's complement addition.**
3. **The sum is complemented and becomes the new checksum.**
4. **If the value of checksum is 0, the message is accepted; otherwise, it is rejected.**